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Membrane shape equations

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Abstract

Shape equations allow the understanding of complex conformations of membranes of cells and cell organelles. We describe the theoretical approaches used to describe the elastic behaviour of lipid membranes, review the sets of equations proposed previously to analyse membrane mechanical equilibrium and discuss their limitations. We further present a derivation of generalized shape equations, which are not limited by any assumptions about the membrane structure and shape. These equations represent a tool for the analysis of complex shapes of cell membranes.

1. Introduction

Biological membranes form cellular borders. Plasma membrane separates cytoplasm from the extracellular medium, hence, determining the cell as an entity. Intracellular membranes serve as boundaries of cell organelles such as endoplasmic reticulum (ER), Golgi Complex (GC), vacuoles, and form nano-compartments operating as intracellular traffic vehicles [1].

The basic element of a biological membrane is a lipid bilayer a few nanometres thick. The membrane dimensions measured along the membrane surface vary from hundreds of nanometres for the intracellular transport carriers to tens of microns for the plasma membranes. Hence, in most biologically relevant cases the membrane width exceeds the thickness by several orders of magnitude and the membrane can be seen as thin film.

Based on the dual hydrophilic–hydrophobic nature of phospholipids molecules, the formation of a membrane is driven by the powerful hydrophobic effect [2], which guarantees stability of the film against ruptures and structural defects. At the same time, the membranes are flexible with respect to bending deformations and, therefore, can change their shapes upon application of weak forces developed by intracellular systems such as a cytoskeleton and/or various proteins binding to the membrane surface [3]. Membrane shapes represent an important biological issue (see for review [3]). Diverse cells have evolved different shapes of plasma membranes such as the corkscrew shape of spirochetes, biconcave disc-like shape of erythrocytes, and flat shape of epithelial cells moving on a substrate. The intracellular organelles and transport carriers have an even larger range of shapes beginning from simple small spheres and thin long cylinders and going up to topologically complex reticular systems.

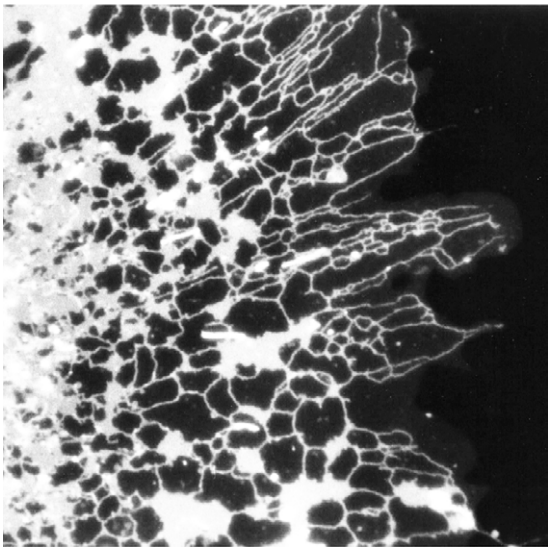

Figure 1. Shape of endoplasmic reticulum of a living COS7 cell tagged with a GFP bound transmembrane protein. Courtesy of K Hirschberg.

Characteristic examples of striking conformations of intracellular organelles are represented by the Golgi Complex (GC) and endoplasmic reticulum (ER) (figure 1) [1]. Geometrically, the shapes of the ER and GC membranes are characterized by two features.

- The membrane curvature changes drastically along the membrane surface.
- The radii of curvature of the interconnected tubes, spheres and discs constituting these organelles equal, approximately, $R \approx 20$–30 nm. This is only several times larger than the membrane thickness $d \approx 4$ nm, which sets a scale for the extent of membrane bending. The bending is considered to be weak (or the curvature to be small) if $d/R \ll 1$, whereas in the case of $d/R \leqslant 1$ the bending is strong (and the curvature is large).

Similar geometrical properties characterize some other intracellular systems such as membrane carriers mediating transport between ER and GC [4–7], mitochondria, and plasma membranes of endothelial cells, which develop internal tubes spanning the cell volume [8, 9].

Summarizing, in many cases the cell membranes are strongly and inhomogeneously curved.

Understanding the origin, stability and dynamics of the peculiar shapes of biological membranes requires analysis of the membrane thermodynamic equilibrium, and, more specifically, of the mechanical equilibrium of membranes with respect to deformations. Such analysis has been performed thoroughly for relatively simple cases of closed lipid bilayers using numerical minimization of the overall energy of membrane deformations (see for review [10–12]). An alternative way to deal with this problem is to solve the equations of membrane mechanical equilibrium called the shape equations. The present paper is devoted to an overview of the membrane shape equations suggested in the literature and the derivation of a generalized set of shape equations allowing analysis of arbitrarily curved membrane shapes, which characterize cell structures.

We proceed as follows. We first present the major geometrical notions necessary to treat membrane shapes. Then we discuss the Gibbs thermodynamic method of membrane

description, mention an alternative approach, which uses the notion of the intra-membrane pressure tensor, and, finally, present the relationships between the two methods. We continue by overviewing the three types of membrane shape equations proposed in the literature during the last three decades, and, finally, present a derivation of a generalized form of these equations.

2. Membrane shape, stresses and mechanical equilibrium

2.1. Geometrical description of a membrane. Dividing surface

Membranes are commonly described as surfaces, which is justified by the above-mentioned smallness of the membrane thickness d compared to the linear dimension $L \sim \sqrt{A}$, where A is the membrane area. Still, because of a non-vanishing thickness d, a systematic approach is required for rigorous determination of the position of the surface used to describe the membrane shape and properties. This issue has been clearly formulated by J W Gibbs, who has introduced the notion of the dividing surface to treat thermodynamics of fluid interfaces [13]. Due to its generality, the Gibbs approach proved to be productive also for the description of lipid membranes, which can be seen as interfaces between aqueous phases. A dividing surface always lies parallel to the membrane plane, but its specific position with respect to the membrane physical elements, such as the layers of the lipid polar heads, can be chosen arbitrarily. At the same time, the major thermodynamic values characterizing the membrane, except for thermodynamic potentials, depend on the position of the dividing surface (see e.g. [14] and references therein). Therefore, once this position has been selected, the whole analysis of membrane shapes must be performed for this specific dividing surface. For symmetry reasons, it is convenient to describe a lipid membrane in terms of the mid-surface lying between the two lipid monolayers forming the lipid bilayer.

A thorough description of the surface shape requires a set of geometrical definitions and relationships, which are given in the appendix. Here we mention only the basic notions necessary for derivation of the thermodynamic relationships and the equations of membrane equilibrium (see e.g. [15–17]).

A surface is characterized at each point by a pair of local covariant basis vectors tangential to the membrane surface and denoted as $\{\vec{r}_\alpha\}$. Here and below the index α and all other Greek indices adopt the values 1 and 2 and indicate two directions tangential to the surface. As, generally, the basis vectors are not mutually orthogonal, one needs also to introduce a set of contravariant basis vectors $\{\vec{r}^\alpha\}$. The unit normal vector to the surface $\vec{n}$ is determined by the vector product of the basis vectors, $\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$. The covariant and contravariant basis vectors give rise to the covariant and contravariant metric tensors denoted as $\{a_{\alpha\beta}\}$ and $\{a^{\alpha\beta}\}$, respectively. The metric tensors perform transformations between the covariant and contravariant components of all vectors and tensors of the surface.

The local basis vectors $\{\vec{r}_\alpha\}$ change from point to point along the surface. Variations of the basis vectors resulting from infinitesimal displacements tangential to the surface are determined by the curvature tensor $\{b^\alpha_\beta\}$ and the Christoffel symbols $\{\Gamma^\lambda_{\alpha\beta}\}$. The two independent invariants of the curvature tensor are the mean curvature, $J = -(b^1_1 + b^2_2)$, and the Gaussian curvature, $K = b^1_1 \cdot b^2_2 - b^2_1 \cdot b^1_2$. Note that, whereas the definition of the total curvature J is convenient for the elastic description of the membranes, the invariant common for mathematical literature is the mean curvature $H = -\frac{1}{2}J$. Alternatively, one use two other invariants of the curvature tensor called the principal curvatures c_1 and c_2, which are given by $c_1 = H + \sqrt{H^2 - K}$ and $c_1 = H - \sqrt{H^2 - K}$.

2.2. Gibbs thermodynamic description of membranes

Within the Gibbs approach, the extensive thermodynamic characteristics of the interface, indicated in the following by superscript S, have a meaning of excess values, which represent differences between these values in the real system and in the system of comparison where the volume phases are extended with unchanged properties up to the dividing surface [13].

Whereas the logic of the Gibbs interface thermodynamics is analogous to that of thermodynamics of bulk phases, it includes three pairs of conjugated variables specific for interfaces. Each pair represents a geometrical characteristic of the dividing surface and the corresponding stress generated within the interface: the surface area A and the lateral tension γ; the total curvature J and the bending moment τ; and the Gaussian curvature K and the stress $\bar{\kappa}$ referred to as the modulus of the Gaussian curvature or the saddle-splay modulus [18].

The major equation of the Gibbs thermodynamics relates the differential of the interface internal energy, $\mathrm{d}U^{\mathrm{S}}$, with the changes of the geometrical characteristics and can be presented is the form [17]

$$\mathrm{d}U^{\mathrm{S}} = T \cdot \mathrm{d}S^{\mathrm{S}} + \sum_i \mu_i \cdot \mathrm{d}m_i^{\mathrm{S}} + \gamma \cdot \mathrm{d}A + \tau \cdot A \cdot \mathrm{d}J + \bar{\kappa} \cdot A \cdot \mathrm{d}K \tag{1}$$

where S^{S} is the surface entropy, the m_i^{S} are the surface excesses of molecules constituting the interface, and the μ_i are the chemical potentials. Note that the original Gibbs representation of this fundamental equation is equivalent to (1) but operates with the principle curvatures c_1 and c_2, and the corresponding stresses rather than with J, τ, and K, $\bar{\kappa}$ [13].

The Gibbs interfacial theory (1) based on the notions of tension and curvature stresses is general and does not involve any assumptions limiting its application. However, using this theory for analysis of particular problems requires further elaboration at the expense of generality of the approach. Specifically, the stress–strain relationships are needed representing γ, τ, and $\bar{\kappa}$ as functions of A, J and K.

2.3. Local thermodynamic approach to description of membranes

A productive model relating the interface stresses to the specific features of interface structure is referred to as the local thermodynamics approach, which was suggested first to treat the surface tension and bending moments of interfaces between fluid phases (see [17] for a review). Later this approach has been used for a description of membranes [19–22]. A membrane is considered as a layer with finite thickness, which can be characterized at each point by a set of thermodynamic variables. As the properties of the membrane change across its thickness, the thermodynamic variables depend on the position inside the membrane. Local interactions within the membrane are effectively characterized by a pressure tensor P_x^y, which is analogous to the usual thermodynamic pressure, but accounts for the possible anisotropy of the intra-membrane forces. The indices of the P_x^y can adopt three values corresponding to one normal (denoted by n) and two tangential (denoted by 1 and 2) directions with respect to the membrane plane:

$$P = \begin{pmatrix} P_n^n & P_1^n & P_2^n \\ P_n^1 & P_1^1 & P_2^1 \\ P_n^2 & P_1^2 & P_2^2 \end{pmatrix}. \tag{2}$$

The index n will be reserved for the normal direction throughout the whole paper. The pressure tensor P_x^y is determined at each point within the membrane and its elements can be expressed in the local basis related to this point. The upper and lower boundaries of the membrane are subjected to the external pressures P_o and P_i, respectively.

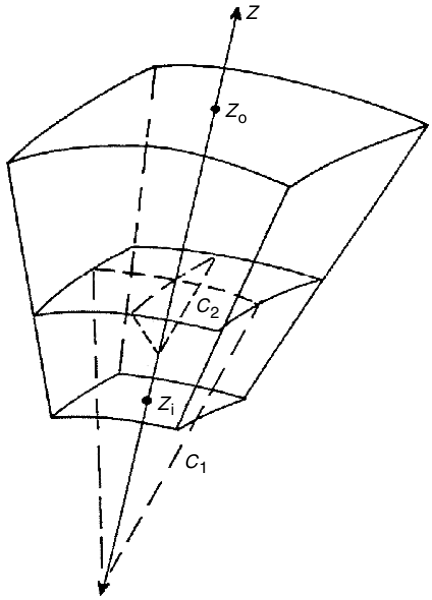

Figure 2. A membrane fragment (notations).

In a simplest case of a homogeneously curved membrane or an interface between immiscible fluid phases (for example, between water and oil), the pressure tensor must have a diagonal form

$$P = \begin{pmatrix} P_n & 0 & 0 \\ 0 & P_T & 0 \\ 0 & 0 & P_T \end{pmatrix}, \tag{3}$$

where the index T denotes the two equivalent tangential directions. The membrane stresses can be expressed in terms of the pressure tensor (3) and the pressures in the bulk. For that the position inside a membrane element is determined by the distance z from the dividing surface measured in the direction perpendicular to the membrane plane (figure 2), whereas the two membrane boundaries are characterized by z_o and z_i. In addition a step-function $P_{i-o}(z)$ has to be defined, which equals P_o above and P_i below the dividing surface. The bending moment τ and the modulus of Gaussian curvature $\bar{\kappa}$ can be expressed in the form [17, 19, 20]

$$\tau = \int_{z_{ii}}^{z_0} (P_{i-o} - P_T) \cdot z \cdot \mathrm{d}z \tag{4}$$

$$\bar{\kappa} = \int_{z_{ii}}^{z_0} (P_{i-o} - P_T) \cdot z^2 \cdot \mathrm{d}z. \tag{5}$$

To determine the lateral tension γ, we first have to define

$$\gamma_0 = \int_{z_{ii}}^{z_0} (P_{i-o} - P_T) \cdot \mathrm{d}z, \tag{6}$$

which gives the value of γ for a flat membrane. The lateral tension of a curved membrane is determined by [17]

$$\gamma = \gamma_0 + \tau \cdot J + \bar{\kappa} \cdot K. \tag{7}$$

2.4. Equations of membrane mechanical equilibrium

The equations of mechanical equilibrium of a membrane can be derived in two ways. The first is minimization of the membrane energy with respect to deformations leading to changes in the membrane area A, total J and Gaussian K curvatures [12, 16, 17, 20]. An alternative way is to consider directly the equilibrium of forces acting on a membrane element and resulting from the membrane tension γ, the bending moment τ, and the Gaussian modulus $\bar{\kappa}$ [19, 23].

Derivation of the equilibrium equations based on minimization of the interface energy (1) was performed by Murphy [17] for liquid interfaces, and confirmed later in a series of related works [24], which addressed fluid interfaces characterized by a diagonal pressure tensor (3). The resulting three equations relate the surface stresses γ, τ and $\bar{\kappa}$ with the curvatures J and K, and describe the equilibrium of the interface element with respect to displacements in the direction normal to the membrane surface:

$$\gamma \cdot J - \tau \cdot (J^2 - 2K) - \bar{\kappa} \cdot J \cdot K = P_i - P_o, \tag{8}$$

displacements in the direction tangential to the membrane surface:

$$\nabla_\alpha \gamma - \tau \cdot \nabla_\alpha J - \bar{\kappa} \cdot \nabla_\alpha K = 0, \tag{9}$$

and, finally, rotation of the interface element with respect to its initial orientation:

$$\nabla_\alpha \tau + J \cdot \nabla_\alpha \bar{\kappa} + b^\nu_\alpha \cdot \nabla_\nu K = 0. \tag{10}$$

In (8)–(10), ∇_α denotes the two-dimensional covariant gradient in the plane of the dividing surface (see the appendix).

Equation (8) for equilibrium in the normal direction represents a generalized form of Laplace equation accounting, in addition to the lateral tension γ, for contributions of bending stresses τ and $\bar{\kappa}$. In the case of small curvatures, neglecting the corresponding terms in (8) results in the traditional form of the Laplace equation, $\gamma \cdot J = \Delta P$, where $\Delta P = P_i - P_o$.

A lipid membrane exhibits properties of a solid film with respect to deformations transversal to its plane, and differs, therefore, from a liquid interface. Minimization of free energy and derivation of equilibrium equation for a lipid membrane have been performed by Helfrich [18] and further developed by Ou-Yang and Helfrich [16] and others (see [12] for references). The Helfrich approach was based on using an elastic model relating the bending moment stresses and the total curvature,

$$\tau = \kappa \cdot (J - J_S), \tag{11}$$

where κ is the bending modulus and J_S is the membrane spontaneous curvature [18]. A major assumption of the Helfrich approach is the smallness to the membrane curvature. The resulting equilibrium equation is

$$\lambda \cdot J + \kappa \cdot (J - J_S) \cdot \left(\tfrac{1}{2} J^2 - 2K + \tfrac{1}{2} J_S \cdot J\right) + \kappa \cdot \nabla^2 J = \Delta P, \tag{12}$$

where λ was called the tensile stress and played a role of Lagrange multiplier conjugated to the membrane area.

The Ou-Yang–Helfrich equation does not assume any particular form of the pressure tensor. Its limitation consists in consideration of small deviations of the membrane shape from the flat one. Therefore, for the cases of small curvature, the Ou-Yang–Helfrich equation is general and includes the Murphy equation (8) as a particular case.

Derivation of the equations of membrane equilibrium based on force balance was performed by Evans and Skalak [23]. This approach was limited neither by addressing fluid interfaces nor by analysis of small curvatures. Its only limitation was related to considerations of axisymmetric membrane shapes. The membrane stresses within this approach were expressed in terms of the tension resultants T_m, T_ϕ and moment resultants, M_m, M_ϕ, where

the subscripts m and ϕ indicate, respectively, the components parallel and perpendicular to the meridian of the axisymmetric shape. The equilibrium equations in the directions tangential and normal to the surface are, respectively,

$$\frac{\partial(rT_m)}{\partial s} - T_\phi \frac{\partial r}{\partial s} + c_m \cdot \left[\frac{\partial(rM_m)}{\partial s} - M_\phi \frac{\partial r}{\partial s} \right] = 0 \tag{13}$$

$$c_m \cdot T_m + c_\phi \cdot T_\phi - \frac{1}{r} \cdot \frac{\partial^2(rM_m)}{\partial s^2} + \frac{1}{r} \cdot \frac{\partial}{\partial s} \left(M_\phi \cdot \frac{\partial r}{\partial s} \right) = \Delta P \tag{14}$$

where c_m and c_ϕ are the principal curvatures of the surface, s is a coordinate measured along the meridian of the shape and r is the radial coordinate measured as a distance from the axis of symmetry.

The tension and moment resultants can be related to the stresses of the Gibbs approach by $T_m = \gamma_0 + \tau \cdot c_\phi$, $T_\phi = \gamma_0 + \tau \cdot c_m$, $M_m = \tau + \bar{\kappa} \cdot c_\phi$, $M_\phi = \tau + \bar{\kappa} \cdot c_m$. Insertion of these relationships into (13) and (14) allows presenting the Evans–Skalak equations in the form

$$\frac{\partial \gamma}{\partial s} - \tau \cdot \frac{\partial J}{\partial s} - \bar{\kappa} \cdot \frac{\partial K}{\partial s} = 0 \tag{15}$$

$$\gamma \cdot J - \tau \cdot (J^2 - 2K) - \bar{\kappa} \cdot J \cdot K - \nabla^2 \tau - c_\phi \cdot \nabla^2 \bar{\kappa} - \frac{\partial \bar{\kappa}}{\partial s} \cdot \frac{\partial c_\phi}{\partial s} = \Delta P \tag{16}$$

where ∇^2 is the two-dimensional Laplace operator in the surface plane. The equation (16) goes beyond the Murphy equation, because it describes membranes, which are non-fluid in the transversal direction and can adopt shapes with inhomogeneous curvatures; it is also more general than the Ou-Yang–Helfrich equation as it is valid for arbitrary rather than small curvatures. At the same time, the Evans–Skalak equations (13)–(16) are limited as compared to the Murphy and Ou-Yang–Helfrich equations by consideration of axisymmetric shapes only.

Summarizing, each of the three previous approaches to derivation of the membrane equilibrium equations is limited by specific assumptions about the membrane structure or extent of bending. They need to be extended to a description of membranes with arbitrary elastic properties and arbitrary shapes. Below we present a derivation of generalized equilibrium equations, which are not limited by any assumptions about the membrane shape or structure of the pressure tensor. We use the approach of force balance with respect to displacements of a membrane element in the normal and tangential directions and with respect to rotation. The resulting equations include as special cases the sets of equilibrium equations mentioned above.

3. Generalized equations of membrane equilibrium

We consider a fragment of a curved membrane, which will be described in terms of a particular dividing surface (figure 2) arbitrarily chosen inside the membrane and called the reference surface. The membrane is considered as consisting of elementary layers parallel to the reference surface. Each elementary layer is characterized by the coordinate z measured along the Z-axis perpendicular to the membrane plane. At the reference surface $z = 0$, while the upper and the lower boundaries of the membrane have the coordinates z_o and z_i, respectively.

The distribution of the internal membrane stresses is determined by the pressure tensor (2) whose elements vary through the membrane thickness, and, hence, depend on the coordinate z.

3.1. Forces applied to the membrane fragment

The total force acting on the membrane fragment consists of the force $\vec{f}$ applied to the sides of the layer, which can be expressed in terms of the pressure tensor P_x^y, and the force $\vec{f}_{\text{out}}$,

which is a balance of forces exerted on the upper and lower boundaries of the membrane and determined by the outer pressures P_o, P_i.

Let us first calculate $\vec{f}$ as a sum of contributions from the elementary layers. The force acting on the sides of an elementary layer and produced only by the tangential components of the pressure tensor is

$$\delta_t \vec{f} = \delta z \cdot \oint \mathrm{d}l \cdot P^{\alpha\beta} \cdot b_\beta \cdot \vec{r}_\alpha \tag{17}$$

where $\mathrm{d}l$ is an element of perimeter of the elementary layer, b_β is a component of the unit vector $\vec{b}$ lying in the plane of the elementary layer perpendicularly to its circumference, and δz is the thickness of the elementary layer. In (17) and throughout the paper we take a sum over the pairs of equal superscripts and subscripts. The integration is performed over the perimeter of the elementary layer taking into account that the basis vectors $\vec{r}_\alpha$ and the components of the pressure tensor $P^{\alpha\beta}$ depend on the position along the perimeter. The differential geometrical relationships used below are presented in the appendix and their numbers are preceded by A.

Accounting for the variations of the basis vectors given by (A.10), applying the Stokes theorem, and retaining only the terms of the first non-vanishing order in the area δs of the element of the dividing surface, we obtain

$$\delta_t \vec{f} = \delta z \cdot \delta s \cdot \left(\vec{r}_\beta \cdot \nabla_\gamma P^{\beta\gamma} + \vec{n} \cdot b_{\lambda\alpha} \cdot P^{\alpha\lambda}\right), \tag{18}$$

where ∇_γ is a component of the covariant gradient, and $b_{\lambda\alpha}$ is a covariant component of the tensor of curvature, both determined at the surface describing the elementary layer.

The contribution to this force of the normal components of the pressure tensor is

$$\delta_n \vec{f} = \delta z \cdot \oint \mathrm{d}l \cdot P^{n\gamma} \cdot b_\gamma \cdot \vec{n}. \tag{19}$$

Analogously to the calculation above, we obtain from (19) in the first non-vanishing order in δs

$$\delta_n \vec{f} = \delta z \cdot \delta s \cdot \left(\vec{n} \cdot \nabla_\lambda P^{n\lambda} - \vec{r}_\nu \cdot b^\nu_\gamma \cdot P^{n\gamma}\right). \tag{20}$$

The total force acting on the sides of the elementary layer is the sum of (18) and (20):

$$\delta \vec{f} = \delta z \cdot \delta s \cdot \left[\vec{r}^{\,\alpha} \cdot \left(\nabla_\lambda P^\lambda_\alpha - b_{\gamma\alpha} \cdot P^{\gamma n}\right) + \vec{n} \cdot \left(b^\alpha_\lambda \cdot P^\lambda_\alpha + \nabla_\gamma P^{\gamma n}\right)\right]. \tag{21}$$

The total force acting on the sides of the membrane fragment is given by integration of (21) over the membrane thickness z. Before performing this integration, we express all the values entering (21) through the characteristics of the reference surface indicated by the front superscript 'o' (see (A.21)–(A.36)). The resulting expressions for the total force acting on the membrane fragment in the tangential direction is

$$f_\alpha = {}^o\mathrm{d}s \cdot \int_{z_i}^{z_o} \mathrm{d}z \left\{{}^o\nabla_\gamma \left[\left(1 + {}^oJ \cdot z\right) \cdot {}^oP^\gamma_\alpha + z \cdot {}^ob^\gamma_\nu \cdot {}^oP^\nu_\alpha\right] - {}^ob^\gamma_\alpha \cdot {}^oP^n_\gamma + z \cdot {}^oK \cdot {}^oP^n_\alpha\right\}. \tag{22}$$

The total force acting on the sides of the membrane fragment in the normal direction is

$$f^n = {}^o\mathrm{d}s \cdot \int_{z_i}^{z_o} \mathrm{d}z \left\{{}^ob^\gamma_\alpha \cdot {}^oP^\alpha_\gamma - z \cdot {}^oK \cdot {}^oP^\alpha_\alpha + {}^o\nabla_\gamma \left[\left(1 + {}^oJ \cdot z\right) \cdot {}^oP^{\gamma n} + z \cdot {}^ob^\gamma_\nu \cdot {}^oP^{\nu n}\right]\right\}. \tag{23}$$

In (22), (23) ${}^o\mathrm{d}s$ is the area of reference surface of the membrane fragment.

The force $\vec{f}_{\mathrm{out}}$ exerted on the membrane fragment by the outside medium has only one component, which is normal to the reference surface and equals

$$f^n_{\mathrm{out}} = {}^o\mathrm{d}s \cdot \left[P_o \cdot \left(1 + z_o \cdot {}^oJ + z_o^2 \cdot {}^oK\right) - P_i \cdot \left(1 + z_i \cdot {}^oJ + z_i^2 \cdot {}^oK\right)\right].$$

The latter relationship can be presented in the form

$$f^n_{\text{out}} = {}^o\delta s \cdot \left[(P_o - P_i) + \int_{z_i}^{z_o} \mathrm{d}z \left({}^oJ + 2 \cdot {}^oK \cdot z \right) \right]. \tag{24}$$

where P_{i-o} is the step function defined above.

3.2. Total torque applied to the membrane fragment

The torque applied to the sides of the membrane fragment will be determined with respect to an arbitrary point at the reference surface, which does not limit the generality of the consideration. The total torque is a sum of contributions of the elementary layers. Similarly to the calculations of the forces above, we determine separately the torque $\delta_t \vec{M}$ exerted on an elementary layer by the tangential components of the pressure tensor, and $\delta_n \vec{M}$, which is due to the normal components of the pressure tensor. The first contribution is given by

$$\delta_t \vec{M} = \delta z \cdot \oint \mathrm{d}l \cdot P^{\alpha\gamma} \cdot b_\gamma \cdot \left[\vec{\rho} \times \vec{r}_\alpha \right], \tag{25}$$

where the integration is performed over the perimeter of the elementary layer, $\vec{\rho}$ is a vector connecting the fixed point at the reference surface and the current point at the perimeter of the elementary layer, and $\times$ denotes the vector product.

The second contribution is

$$\delta^n \vec{M} = \delta z \cdot \oint \mathrm{d}l \cdot P^{n\gamma} \cdot b_\gamma \cdot \left[\vec{\rho} \times \vec{n} \right]. \tag{26}$$

The following calculation of the total torque is based on (25), (26) and is analogous to the above determination of the total force. Accounting for the expressions (A.10), (A.11) derived in the appendix, we apply the Stokes theorem to (25), (26), retain the terms of the first non-vanishing order in the area of the membrane fragment and express all the values through the characteristics of the reference surface. As a result, the tangential component of the torque (corresponding to rotation of the vector normal to the reference surface) is

$$M^\alpha = {}^o\mathrm{d}s \cdot \int_{z_i}^{z_o} \mathrm{d}z \cdot {}^oc^{\beta\alpha} \cdot \left\{ z \cdot {}^o\nabla^o_\nu P^\nu_\beta + z^2 \cdot {}^o\nabla_\nu \left({}^oJ \cdot {}^oP^\nu_\beta + {}^ob^\nu_\gamma \cdot {}^oP^\gamma_\beta \right) \right.$$
$$\left. - \left({}^oP^n_\beta + z \cdot {}^oJ \cdot {}^oP^n_\beta + z \cdot {}^ob^\gamma_\beta \cdot {}^oP^n_\gamma \right) \right\}, \tag{27}$$

where ${}^oc^{\beta\alpha}$ is the discriminant tensor at the reference surface (A.6). In the following we drop the superscript 'o' keeping in mind that all the values are related to the reference surface.

3.3. Equations of equilibrium

To present the equations of equilibrium in the most compact form, we first define the following tensors:

$$C^\nu_{0\beta} = \int_{z_i}^{z_o} \mathrm{d}z \cdot \left(P_{i-o} \cdot \delta^\nu_\beta - P^\nu_\beta \right), \tag{28}$$

$$C^\nu_{1\beta} = \int_{z_i}^{z_o} \mathrm{d}z \cdot z \cdot \left(P_{i-o} \cdot \delta^\nu_\beta - P^\nu_\beta \right), \tag{29}$$

$$C^\nu_{2\beta} = \int_{z_i}^{z_o} \mathrm{d}z \cdot z^2 \cdot \left(P_{i-o} \cdot \delta^\nu_\beta - P^\nu_\beta \right), \tag{30}$$

where δ^ν_β is the unitary tensor. The tensors (28)–(30) will be referred to as the tensors of zero, first and second bending moments, respectively.

The equation of equilibrium in the normal direction follows from condition of zero total normal force given by the sum of (23) and (24). Accounting for (23), (24) and (28)–(30), this condition can be presented in the form

$$b^{\alpha}_{\lambda}\cdot C^{\lambda}_{0\alpha}-K\cdot C^{\lambda}_{1\lambda}-\nabla_{\gamma}\cdot\int_{z_i}^{z_o}\mathrm{d}z\cdot\left[(1+J\cdot z)\cdot P^{n\gamma}+z\cdot b^{\gamma}_{\nu}\cdot P^{n\nu}\right]=P_o-P_i. \tag{31}$$

To express the equations of zero lateral force in terms of the moments (28)–(30) we note that the tensor $P_{i-o}\delta^{\nu}_{\beta}$ is constant over the surface. Therefore, the terms proportional to the gradient of this tensor are zero and can be added to the expression (22) without changing the latter. This results in

$$\nabla_{\gamma}C^{\gamma}_{0\alpha}+J\cdot\nabla_{\gamma}C^{\gamma}_{1\alpha}+b^{\gamma}_{\nu}\cdot\nabla_{\gamma}C^{\nu}_{1\alpha}+b^{\gamma}_{\alpha}\cdot\int_{z_i}^{z_o}P^{n}_{\gamma}\,\mathrm{d}z-K\cdot\int_{z_i}^{z_o}z\cdot P^{n}_{\gamma}\,\mathrm{d}z=0. \tag{32}$$

The equation for zero torque following from (27) is

$$\nabla_{\gamma}C^{\gamma}_{1\alpha}+J\cdot\nabla_{\gamma}C^{\gamma}_{2\alpha}+b^{\gamma}_{\nu}\cdot\nabla_{\gamma}C^{\nu}_{2\alpha}+\int_{z_i}^{z_o}[(1+z\cdot J)\cdot P^{n}_{\alpha}+z\cdot b^{\nu}_{\alpha}\cdot P^{n}_{\nu}]\,\mathrm{d}z=0. \tag{33}$$

The equilibrium equations (31), (32) can be rewritten in a different form where the contributions of the normal components of the pressure tensor $P^{\nu n}$ are expressed, using (33), in terms of the bending moments.

Then the equation of equilibrium in the normal direction has the form

$$b^{\alpha}_{\lambda}\cdot C^{\lambda}_{0\alpha}-K\cdot C^{\lambda}_{1\lambda}-\nabla^{\alpha}\left(\nabla_{\gamma}C^{\gamma}_{1\alpha}+J\cdot\nabla_{\gamma}C^{\gamma}_{2\alpha}+b^{\gamma}_{\nu}\cdot\nabla_{\gamma}C^{\nu}_{2\alpha}\right)=P_o-P_i, \tag{34}$$

and the equation of equilibrium in the tangential direction is

$$\nabla_{\gamma}C^{\gamma}_{0\alpha}+J\cdot\nabla_{\gamma}C^{\gamma}_{1\alpha}+b^{\gamma}_{\nu}\cdot\nabla_{\gamma}C^{\nu}_{1\alpha}-b^{\gamma}_{\alpha}\cdot\nabla_{\nu}C^{\nu}_{1\gamma}-J\cdot b^{\beta}_{\alpha}\cdot\nabla_{\nu}C^{\nu}_{1\beta}-b^{\beta}_{\alpha}\cdot b^{\nu}_{\gamma}\cdot\nabla_{\nu}C^{\gamma}_{2\beta}=0. \tag{35}$$

3.4. Equilibrium equations in special cases

The equilibrium equations (31)–(35) derived above are valid in a general case of a non-diagonal pressure tensor. Therefore, they describe interfaces of arbitrary structure including layers, which have properties of solid films in both tangential and normal directions. The equilibrium equations are simplified if the interface possesses properties of a fluid at least in some of the directions.

The simplest case is that of a completely fluid interface such as those between two immiscible liquids. As mentioned above, for such systems the pressure tensor has a diagonal form (3) independently of the shape of the interface, meaning that its components P^{β}_{γ} for $\gamma\neq\beta$, and P^{n}_{γ} vanish. In this case the tensors of the bending moments (28)–(30) have the forms

$$C^{\alpha}_{0\beta}=C_0\cdot\delta^{\alpha}_{\beta},\qquad C^{\alpha}_{1\beta}=C_1\cdot\delta^{\alpha}_{\beta},\qquad C^{\alpha}_{2\beta}=C_2\cdot\delta^{\alpha}_{\beta}, \tag{36}$$

where C_0, C_1, and C_2 represent, respectively, the stress factor γ_0, the bending moment τ, and the modulus of Gaussian curvature $\bar{\kappa}$, given by the equations (6), (4) and (5).

Inserting (36) into (31)–(35), taking into account that P^{n}_{γ} vanish, and using (A.14) we obtain for equilibrium in the normal direction:

$$J\cdot\gamma_0+2\cdot K\cdot\tau=P_i-P_0, \tag{37}$$

for equilibrium in the tangential direction:

$$\nabla_{\gamma}\gamma_0+J\cdot\nabla_{\gamma}\tau+K\cdot\nabla_{\gamma}\bar{\kappa}=0, \tag{38}$$

and for equilibrium with respect to rotation:

$$\nabla_\alpha \tau + J \cdot \nabla_\alpha \bar{\kappa} + b^\nu_\alpha \cdot \nabla_\nu K = 0. \tag{39}$$

Inserting into (37)–(39) the expression (7) of the stress factor γ_0 through τ, and $\bar{\kappa}$, we obtain the Murphy equations.

Consider a lipid membrane, which has properties of a liquid in the tangential direction, but retains the properties of a solid film in the normal direction. In this case, the normal components of the pressure tensor P^n_γ can be different from zero, while the non-diagonal tangential components (P^α_β for $\alpha \neq \beta$) vanish independently of the membrane shape. Inserting (36) into (33)–(35) and simplifying them we find the equation of equilibrium in the normal direction

$$J \cdot \gamma_0 + 2 \cdot K \cdot \tau + \nabla^\beta \left(\nabla_\beta \tau + J \cdot \nabla_\beta \bar{\kappa} + b^\gamma_\beta \nabla_\gamma \bar{\kappa} \right) = P_i - P_o, \tag{40}$$

the equation of equilibrium in the tangential direction having the form (38), and the equation of equilibrium with respect to rotation

$$\nabla_\alpha \tau + J \cdot \nabla_\alpha \bar{\kappa} + b^\nu_\alpha \cdot \nabla_\nu K + \int_{z_i}^{z_o} [(1 + z \cdot J) \cdot P^n_\alpha + z \cdot b^\nu_\alpha \cdot P^n_\nu] \, \mathrm{d}z = 0. \tag{41}$$

The equations (40) and (38) can be reduced to the Ou-Yang–Helfrich shape equation in a particular case where the bending moment τ is related to the total curvature according to (11), whereas the modulus of Gaussian curvature $\bar{\kappa}$ is assumed to be constant along the membrane.

Summarizing, the equilibrium equations (31)–(35) derived in this section have a general character going beyond any specific model and are valid for membranes and interfaces with arbitrary structures and shapes. In special cases these equations provide the previously suggested sets of equilibrium equations.

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Appendix. Summary of geometrical definitions, relationships and derivations

Consider within the membrane fragment an arbitrary dividing surface determined by vector $\vec{R}(\xi^1, \xi^2)$. At each point of the surface a pair of independent covariant tangential basis vectors $\{\vec{r}_\alpha\}$ are given by

$$\vec{r}_\alpha = \frac{\partial \vec{R}}{\partial \xi^\alpha}. \tag{A.1}$$

The contravariant tangential basis vectors $\{\vec{r}^\alpha\}$ are determined according to the condition that their scalar products with the covariant basis vectors are given by the symmetric unitary tensor δ^β_α,

$$\vec{r}_\alpha \cdot \vec{r}^\beta = \delta^\beta_\alpha. \tag{A.2}$$

The elements of the covariant and contravariant metric tensor are determined by the scalar products of the corresponding basis vectors

$$a_{\alpha\beta} = \vec{r}_\alpha \cdot \vec{r}_\beta, \qquad a^{\alpha\beta} = \vec{r}^\alpha \cdot \vec{r}^\beta. \tag{A.3}$$

An element of surface area is determined by

$$\mathrm{d}s = \sqrt{a} \cdot \mathrm{d}\xi^1 \mathrm{d}\xi^2, \tag{A.4}$$

where $a = \det(a_{\alpha\beta})$.

The unit normal to the surface is

$$\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{\sqrt{a}}. \tag{A.5}$$

For applications we define the covariant discriminant tensor

$$c_{\alpha\beta} = \sqrt{a} \cdot \varepsilon_{\alpha\beta}, \tag{A.6}$$

where $\varepsilon_{11} = \varepsilon_{22} = 0$, whereas $\varepsilon_{12} = \varepsilon_{21} = -1$.

Consider the derivatives of the basis vectors with respect to displacements along the surface, $\vec{r}_{\alpha\beta} = \frac{\partial \vec{r}_\alpha}{\partial \xi^\beta}$. They are determined by the Gauss formulae

$$\vec{r}_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} \cdot \vec{r}_\nu + b_{\alpha\beta} \cdot \vec{n}, \tag{A.7}$$

where $\Gamma^\nu_{\alpha\beta}$ are the Christoffel symbols, and $b_{\alpha\beta}$ is the covariant tensor of curvature, which can be also presented as a scalar product

$$b_{\alpha\beta} = \vec{n} \cdot \vec{r}_{\alpha\beta} = -\vec{n}_\alpha \cdot \vec{n}_\beta. \tag{A.8}$$

Transitions between the covariant and contravariant components of the vectors and tensors is performed by means of the metric tensor, e.g.,

$$b^\alpha_\beta = a^{\alpha\gamma} \cdot b_{\gamma\beta}. \tag{A.9}$$

As noted in the body of the paper, the sum is taken over each pair of equal indices.

Using (A.7), (A.9) we can determine the infinitesimal changes along the surface of the basis vectors and the unit normal vector

$$\vec{r}_\alpha \left(\left\{ \xi^\beta + \mathrm{d}\xi^\beta \right\} \right) = \vec{r}_\alpha + \left[\Gamma^\nu_{\alpha\beta} \cdot \vec{r}_\nu + b_{\alpha\beta} \cdot \vec{n} \right] \cdot \mathrm{d}\xi^\beta, \tag{A.10}$$

$$\vec{n} \left(\left\{ \xi^\beta + \mathrm{d}\xi^\beta \right\} \right) = \vec{n} - b^\nu_\gamma \cdot \vec{r}_\nu \cdot \mathrm{d}\xi^\gamma. \tag{A.11}$$

The two independent scalars of the curvature tensor are the total curvature

$$J = -b^\alpha_\alpha = -b^\nu_\alpha \cdot \delta^\alpha_\nu, \tag{A.12}$$

and the Gaussian curvature

$$K = \tfrac{1}{2} \cdot c_{\alpha\beta} \cdot c^{\lambda\gamma} \cdot b^\alpha_\lambda \cdot b^\beta_\gamma. \tag{A.13}$$

Useful relationships between the components of the curvature tensor, which can be readily proved, are

$$b^\alpha_\lambda \cdot b^\lambda_\beta = -J \cdot b^\alpha_\beta - K \cdot \delta^\alpha_\beta \tag{A.14}$$

$$b^\beta_\lambda \cdot b^\lambda_\beta = J^2 - 2K. \tag{A.15}$$

The covariant derivative (gradient) of a scalar is

$$\nabla_\alpha \varphi = \frac{\partial \varphi}{\partial \xi^\alpha}, \tag{A.16}$$

of a vector surface $\vec{A}$ is

$$\nabla_\beta A_\alpha = \frac{\partial A_\alpha}{\partial \xi^\beta} - \Gamma^\lambda_{\alpha\beta} \cdot A_\lambda, \tag{A.17}$$

and of a tensor is

$$\nabla_\gamma f_{\alpha\beta} = \frac{\partial f_{\alpha\beta}}{\partial \xi^\gamma} - \Gamma^\lambda_{\gamma\alpha} \cdot f_{\lambda\beta} - \Gamma^\lambda_{\gamma\beta} \cdot f_{\alpha\lambda}. \tag{A.18}$$

The derivatives of contravariant and mixed components of surface vectors and tensors can be derived from (A.16)–(A.18).

The covariant divergence of a vector is

$$\operatorname{div}\vec{A} = \nabla_\alpha A^\alpha = \frac{1}{\sqrt{a}} \cdot \frac{\partial\left(\sqrt{a}A^\alpha\right)}{\partial\xi^\alpha}. \tag{A.19}$$

The Stokes theorem relating the integral of a surface vector $\vec{A}$ over a closed contour L on the surface to integral of the divergence of $\vec{A}$ over the area S limited by this contour is expressed as

$$\oint_L A^\alpha \cdot b_\alpha \cdot \mathrm{d}l = \int_S \operatorname{div}\vec{A} \cdot \mathrm{d}s, \tag{A.20}$$

where b_α is a unit vector tangential to the surface and normal to the contour.

In the following, we present the relationships between the geometrical characteristics of the reference surface and those of a surface representing an arbitrary elementary layer of the membrane fragment. The two surfaces are parallel with distance z between them. The values related to the reference surface will be denoted by the front superscript 'o'.

The relation between the vectors determining the two surfaces is

$$\vec{R} = {}^o\vec{R} + {}^o\vec{n} \cdot z. \tag{A.21}$$

Based on (A.21), we obtain the relationships for the covariant and contravariant basis vectors

$$\vec{r}_\alpha = {}^o\vec{r}_\alpha - {}^o b_\alpha^\gamma \cdot {}^o\vec{r}_\gamma \cdot z, \tag{A.22}$$

$$\vec{r}^\alpha = \frac{(1 + {}^oJ \cdot z) \cdot {}^o\vec{r}^\alpha + {}^o b_\gamma^\alpha \cdot {}^o\vec{r}^\gamma \cdot z}{1 + {}^oJ \cdot z + {}^oK \cdot z^2}; \tag{A.23}$$

for the covariant metric tensor

$$a_{\alpha\beta} = (1 - {}^oK \cdot z^2) \cdot {}^o a_{\alpha\beta} - (2 \cdot z + {}^oJ \cdot z^2) \cdot {}^o b_{\alpha\beta}; \tag{A.24}$$

for the contravariant metric tensor

$$a^{\alpha\beta} = \frac{(1 + 2 \cdot {}^0J \cdot z + {}^0J^2 \cdot z^2 - {}^0K \cdot z^2) \cdot {}^0a^{\alpha\beta} + (2 \cdot z + {}^0J \cdot z^2) \cdot {}^0b^{\alpha\beta}}{\left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)^2}; \tag{A.25}$$

for the determinant of the metric tensor

$$a = \left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)^2 \cdot {}^0a; \tag{A.26}$$

for the covariant curvature tensor

$$b_{\alpha\beta} = (1 + {}^0J \cdot z) \cdot {}^0b_{\alpha\beta} + {}^0K \cdot {}^0a_{\alpha\beta} \cdot z; \tag{A.27}$$

for the mixed tensor of curvature

$$b_\beta^\alpha = \frac{{}^0b_\beta^\alpha - {}^0K \cdot \delta_\beta^\alpha \cdot z}{\left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)}; \tag{A.28}$$

for the covariant discriminant tensor

$$c_{\alpha\beta} = \left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right) \cdot {}^0c_{\alpha\beta} \tag{A.29}$$

for the contravariant discriminant tensor

$$c^{\alpha\beta} = \frac{{}^0c^{\alpha\beta}}{\left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)}; \tag{A.30}$$

for the Christoffel symbols

$$\Gamma_{\alpha\beta}^\lambda = {}^0\Gamma_{\alpha\beta}^\lambda - z \cdot \frac{\left(1 - {}^0J \cdot z\right) \cdot {}^0\nabla_\beta {}^0b_\alpha^\lambda + z \cdot {}^0b_\nu^\lambda \cdot {}^0\nabla_\beta {}^0b_\alpha^\nu}{\left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)}; \tag{A.31}$$

for the area element

$$ds = \left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right) \cdot {}^0ds; \tag{A.32}$$

for the total curvature

$$J = \frac{{}^0J + 2 \cdot {}^0K \cdot z}{\left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)}; \tag{A.33}$$

for the Gaussian curvature

$$K = \frac{{}^0K}{\left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)}; \tag{A.34}$$

and for the components of the pressure tensor

$$P_\alpha^\gamma = \left({}^0P_\alpha^\gamma + z \cdot \left({}^0J + {}^0b_\nu^\gamma \cdot {}^0P_\alpha^\nu - {}^0b_\alpha^\lambda \cdot {}^0P_\lambda^\gamma\right) - z^2 \cdot \left({}^0J \cdot {}^0b_\alpha^\lambda \cdot {}^0P_\lambda^\gamma - {}^0b_\nu^\gamma \cdot {}^0b_\alpha^\lambda \cdot {}^0P_\lambda^\nu\right)\right) \times \left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)^{-1}, \tag{A.35}$$

$$P^{n\gamma} = \frac{\left(1 + {}^0J \cdot z\right) \cdot {}^0P^{n\gamma} + z \cdot {}^0b_\nu^\gamma \cdot {}^0P^{n\gamma}}{\left(1 + {}^0J \cdot z + {}^0K \cdot z^2\right)}. \tag{A.36}$$

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